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Helicity Basis and Parity

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We study the theory of the (1/2,0) ⊕ (0,1/2) representation in helicity basis. Helicity eigenstates are not the parity eigenstates. This is in accordance with the consideration of Berestetskiĭ, Lifshitz, and Pitaevskiĭ. Relations to the Gelfand-Tsetlin-Sokolik-type quantum field theory are discussed. Finally, a new form of the parity operator is proposed. It commutes with the Hamiltonian.

KEY WORDS: helicity basis; parity; Gelfand–Tsetlin–Sokolik-type quantum field theory; Lorentz group representations.

Recently we generalized the Dirac formalism (Ahluwalia, 1996; Barut and Ziino, 1993; Dvoeglazov, 1997a,b, 2000, 2002a; Gupta, 1967; Ziino, 1989, 1991, 1996) and the Bargmann-Wigner formalism (Dvoeglazov, 2001, 2002b), and on this basis we proposed a set of 12 equations for antisymmetric tensor (AST) field; some of them may lead to parity-violating transitions. In this paper we are going to study somewhat related matter, the transformation from the standard basis to the helicity basis in the Dirac theory. The spin basis rotation *changes* the properties of corresponding states with respect to parity. The parity is a physical quantum number; so, we try to extract corresponding physical contents from considerations of the various spin bases.

Briefly, I repeat the results of Dvoeglazov (2002b,c). One can find solutions of the $2(2J+1)$-theory with different parity properties (Dvoeglazov, 2002c). They can be related to the polarization vectors obtained by Ruck and Greiner (1977), who found the helicity states of the 4-vector potential on the basis of the Jackob and Wick paper (Jackob and Wick, 1959). Next, I used the generalized Bargmann-Wigner formalism based on the equations[2]

$$[i\gamma_\mu\partial_\mu + \epsilon_1 m_1 + \epsilon_2 m_2\gamma_5]_{\alpha\beta}\Psi_{\beta\gamma} = 0, \tag{1a}$$

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[2] The parity-violating Dirac equation has been derived in (Dvoeglazov, 2000, 2002a). The method of the derivation refers to the van der Waerden, Sakurai, and Gersten works, see references in the previous papers of mine.

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$$[i\gamma_\mu \partial_\mu + \epsilon_3 m_1 + \epsilon_4 m_2 \gamma_5]_{\alpha\beta} \Psi_{\gamma\beta} = 0, \tag{1b}$$

Different equations for the antisymmetric tensor field follow from this set by means of the standard procedure (Luriè, 1968). We concluded in Dvoeglazov (2002b) in part that (1) in the (1/2, 0) ⊕ (0, 1/2) representation it is possible to introduce the *parity-violating* frameworks; (2) the mappings between the Weinberg–Tucker–Hammer formalism for $J = 1$ and the AST fields of the 2nd rank of, at least, eight types exist; four of them include both $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, which tells us that the *parity violation* may occur during the study of the corresponding dynamics; (3) if we want to take into account the $J = 1$ solutions with different parity properties, the Bargmann–Wigner (BW) formalism is to be generalized; (4) the 4-potentials and the fields in the helicity basis can be constructed; they have different parity properties comparing with the standard ("parity") basis; (5) generalizing the BW formalism in such a way, 12 equations for the AST fields have been obtained; (6) finally, a hypothesis was proposed therein that the obtained results are related to the spin basis rotations and to the choice of normalization.

Beginning the consideration of the helicity basis, we observe that it is well known that the operator $\hat{\mathbf{S}}_3 = \boldsymbol{\sigma}_3/2 \otimes I_2$ does not commute with the Dirac Hamiltonian unless the 3-momentum is aligned along with the third axis and the plane-wave expansion is used:

$$[\hat{\mathcal{H}}, \hat{\mathbf{S}}_3]_- = (\gamma^0 \gamma^k \times \nabla_i)_3 \tag{2}$$

Moreover, Berestetskiĭ, Lifshtz and Pitaevskiĭ wrote (Berestetskiĭ *et al.*, 1982): "... the orbital angular momentum **l** and the spin **s** of a moving particle are not separately conserved. Only the total angular momentum $\mathbf{j} = \mathbf{l} + \mathbf{s}$ is conserved. The component of the spin in any fixed direction (taken as z-axis is therefore also not conserved, and cannot be used to enumerate the polarization (spin) states of moving particle." The similar conclusion has been given by Novozhilov in his book (Novozhilov, 1975). On the other hand, the helicity operator $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}/2 \otimes I$, $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, commutes with the Hamiltonian (more precisely, the commutator is equal to zero when acting the one-particle plane-wave solutions).

So, it is a bit surprising, why the 4-spinors have been studied so well when the basis was chosen in such a way that they are eigenstates of the $\hat{\mathbf{S}}_3$ operator:

$$u_{\frac{1}{2},\frac{1}{2}} = N^+_{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_{\frac{1}{2},-\frac{1}{2}} = N^+_{-\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$$v_{\frac{1}{2},\frac{1}{2}} = N^-_{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad v_{\frac{1}{2},-\frac{1}{2}} = N^-_{\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{3}$$

and, oppositely, the helicity basis case has not been studied almost at all (see, however, Jackob and Wick, 1959; Novozhilov, 1975). Let me remind that the boosted 4-spinors in the "common-used" basis are

$$u_{\frac{1}{2},\frac{1}{2}} = \frac{N^{+}_{\frac{1}{2}}}{\sqrt{2m(E+m)}}\begin{pmatrix} p^{+}+m \\ p_r \\ p^{-}+m \\ -p_r \end{pmatrix},$$

$$u_{\frac{1}{2},-\frac{1}{2}} = \frac{N^{+}_{-\frac{1}{2}}}{\sqrt{2m(E+m)}}\begin{pmatrix} p_l \\ p^{-}+m \\ -p_l \\ p^{+}+m \end{pmatrix}, \tag{4a}$$

$$v_{\frac{1}{2},\frac{1}{2}} = \frac{N^{-}_{\frac{1}{2}}}{\sqrt{2m(E+m)}}\begin{pmatrix} p^{+}+m \\ p_r \\ -p^{-}-m \\ p_r \end{pmatrix},$$

$$v_{\frac{1}{2},-\frac{1}{2}} = \frac{N^{-}_{-\frac{1}{2}}}{\sqrt{2m(E+m)}}\begin{pmatrix} p_l \\ p^{-}+m \\ p_l \\ -p^{+}-m \end{pmatrix}, \tag{4b}$$

$p^{\pm} = E \pm p_z$, $p_{r,l} = p_x \pm i p_y$. They are the parity eigenstates with eigenvalues of ± 1. In the parity operator the matrix

$$\left\{ \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\}$$

is used.

Let me turn now your attention to the helicity spin basis. The 2-eigenspinors of the helicity operator

$$\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{p}} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{+i\phi} & -\cos\theta \end{pmatrix} \tag{5}$$

can be defined as follows (Varshalovich *et al.*, 1988; Dvoeglazov, 1997c):

$$\phi_{\frac{1}{2}\uparrow} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{+i\phi/2} \end{pmatrix}, \quad \phi_{\frac{1}{2}\downarrow} = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\cos\frac{\theta}{2}\, e^{+i\phi/2} \end{pmatrix}, \tag{6}$$

for $\pm 1/2$ eigenvalues, respectively.

We start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i. e., two degrees of freedom); $c = \hbar = 1$:

$$(E + \boldsymbol{\sigma} \cdot \mathbf{p})(E - \boldsymbol{\sigma} \cdot \mathbf{p})\phi = m^2\phi. \tag{7}$$

It can be rewritten in the form of the set of two first-order equations for 2-spinors. Simultaneously, we observe that they may be chosen as eigenstates of the helicity operator which is present in (7):[3]

$$(E - (\boldsymbol{\sigma} \cdot \mathbf{p}))\phi_\uparrow = (E - p)\phi_\uparrow = m\chi_\uparrow, \tag{8a}$$

$$(E + (\boldsymbol{\sigma} \cdot \mathbf{p}))\chi_\uparrow = (E + p)\chi_\uparrow = m\phi_\uparrow, \tag{8b}$$

$$(E - (\boldsymbol{\sigma} \cdot \mathbf{p}))\phi_\downarrow = (E + p)\phi_\downarrow = m\chi_\downarrow, \tag{8c}$$

$$(E + (\boldsymbol{\sigma} \cdot \mathbf{p}))\chi_\downarrow = (E - p)\chi_\downarrow = m\phi_\downarrow. \tag{8d}$$

If the ϕ spinors are defined by the equation (6) then we can construct the corresponding u- and v-4-spinors[4]

$$u_\uparrow = N_\uparrow^+ \begin{pmatrix} \phi_\uparrow \\ \dfrac{E-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\dfrac{E+p}{m}}\phi_\uparrow \\ \sqrt{\dfrac{m}{E+p}}\phi_\uparrow \end{pmatrix},$$

$$u_\downarrow = N_\downarrow^+ \begin{pmatrix} \phi_\downarrow \\ \dfrac{E+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\dfrac{m}{E+p}}\phi_\downarrow \\ \sqrt{\dfrac{E+p}{m}}\phi_\downarrow \end{pmatrix}, \tag{10a}$$

[3] This opposes to the choice of the basis (3), where 4-spinors are the eigenstates of the parity operator.

[4] One can also try to construct yet another theory differing from the ordinary Dirac theory. The 4-spinors might be not the eigenspinors of the helicity operator of the $(1/2, 0) \oplus (0, 1/2)$ representation space (cf. Ahluwalia, 1996; Dvoeglazov, 1997a; Gupta, 1967). They might be the eigenstates of the *chiral* helicity operator introduced in Gupta (1967). In this case, the momentum–space Dirac equations can be written (cf. Dvoeglazov, 1997a,b)

$$p_\mu\gamma^\mu\mathcal{U}_\uparrow - m\mathcal{U}_\downarrow = 0, \tag{9a}$$

$$p_\mu\gamma^\mu\mathcal{U}_\downarrow - m\mathcal{U}_\uparrow = 0, \tag{9b}$$

$$p_\mu\gamma^\mu\mathcal{V}_\uparrow + m\mathcal{V}_\downarrow = 0, \tag{9c}$$

$$p_\mu\gamma^\mu\mathcal{V}_\downarrow + m\mathcal{V}_\uparrow = 0. \tag{9d}$$

Here $\uparrow\downarrow$ refers already to the chiral helicity eigenstates, e.g. $u_\eta = \frac{1}{\sqrt{2}}\begin{pmatrix} N\phi_\eta \\ N^{-1}\phi_{-\eta} \end{pmatrix}$.

$$v_\uparrow = N_\uparrow^- \begin{pmatrix} \phi_\uparrow \\ -\dfrac{E-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\dfrac{E+p}{m}}\phi_\uparrow \\ -\sqrt{\dfrac{m}{E+p}}\phi_\uparrow \end{pmatrix},$$

$$v_\downarrow = N_\downarrow^- \begin{pmatrix} \phi_\downarrow \\ -\dfrac{E+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\dfrac{m}{E+p}}\phi_\downarrow \\ -\sqrt{\dfrac{E+p}{m}}\phi_\downarrow \end{pmatrix}, \qquad (10b)$$

where the normalization to the unit (± 1) was used:[5]

$$\bar{u}_\lambda u_{\lambda'} = \delta_{\lambda\lambda'}, \quad \bar{v}_\lambda v_{\lambda'} = -\delta_{\lambda\lambda'}, \qquad (11a)$$

$$\bar{u}_{\lambda'} v_{\lambda'} = 0 = \bar{v}_\lambda u_{\lambda'} \qquad (11b)$$

One can prove that the matrix

$$P = \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad (12)$$

can be used in the parity operator as well as in the original Dirac basis. Indeed, the 4-spinors (10a,10b) satisfy the Dirac equation in the spinorial representation of the γ-matrices (see straightforwardly from (7)). Hence, the parity-transformed function $\Psi'(t, -\mathbf{x}) = P\Psi(t, \mathbf{x})$ must satisfy

$$[i\gamma^\mu \partial'_\mu - m]\Psi'(t, -\mathbf{x}) = 0, \qquad (13)$$

with $\partial'_\mu = (\partial/\partial t, -\nabla_i)$. This is possible when $P^{-1}\gamma^0 P = \gamma^0$ and $P^{-1}\gamma^i P = -\gamma^i$. The matrix (12) satisfies these requirements, as in the textbook case.

Next, it is easy to prove that one can form the projection operators

$$P_+ = +\sum_\lambda u_\lambda(\mathbf{p})\bar{u}_\lambda(\mathbf{p}) = \frac{p_\mu\gamma^\mu + m}{2m}, \qquad (14a)$$

$$P_- = -\sum_\lambda v_\lambda(\mathbf{p})\bar{v}_\lambda(\mathbf{p}) = \frac{m - p_\mu\gamma^\mu}{2m}, \qquad (14b)$$

with the properties $P_+ + P_- = 1$ and $P_\pm^2 = P_\pm$. This permits us to expand the 4-spinors defined in the basis (3) in linear superpositions of the helicity basis 4-spinors and to find corresponding coefficients of the expansion:

$$u_\sigma(\mathbf{p}) = A_{\sigma\lambda} u_\lambda(\mathbf{p}) + B_{\sigma\lambda} v_\lambda(\mathbf{p}), \qquad (15a)$$

$$v_\sigma(\mathbf{p}) = C_{\sigma\lambda} u_\lambda(\mathbf{p}) + D_{\sigma\lambda} v_\lambda(\mathbf{p}). \qquad (15b)$$

[5] Of course, there are no any mathematical difficulties to change it to the normalization to $\pm m$, which may be more convenient for our study of the massless limit.

Multiplying the above equations by $\bar{u}_{\lambda'}$, $\bar{v}_{\lambda'}$ and using the normalization conditions, we obtain $A_{\sigma\lambda} = D_{\sigma\lambda} = \bar{u}_\lambda u_\sigma$, $B_{\sigma\lambda} = C_{\sigma\lambda} = -\bar{v}_\lambda u_\sigma$. Thus, the transformation matrix from the common-used basis to the helicity basis is

$$\begin{pmatrix} u_\sigma \\ v_\sigma \end{pmatrix} = \mathcal{U} \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{16}$$

Neither A nor B are unitary:

$$A = (a_{++} + a_{+-})(\sigma_\mu a^\mu) + (-a_{-+} + a_{--})(\sigma_\mu a^\mu)\sigma_3, \tag{17a}$$

$$B = (-a_{++} + a_{+-})(\sigma_\mu a^\mu) + (a_{-+} + a_{--})(\sigma_\mu a^\mu)\sigma_3, \tag{17b}$$

where

$$a^0 = -i\cos(\theta/2)\sin(\phi/2) \in \Im m, \quad a^1 = \sin(\theta/2)\cos(\phi/2) \in \Re e, \tag{18a}$$

$$a^2 = \sin(\theta/2)\sin(\phi/2) \in \Re e, \quad a^3 = \cos(\theta/2)\cos(\phi/2) \in \Re e, \tag{18b}$$

and

$$a_{++} = \frac{\sqrt{(E+m)(E+p)}}{2\sqrt{2}m}, \quad a_{+-} = \frac{\sqrt{(E+m)(E-p)}}{2\sqrt{2}m}, \tag{19a}$$

$$a_{-+} = \frac{\sqrt{(E-m)(E+p)}}{2\sqrt{2}m}, \quad a_{--} = \frac{\sqrt{(E-m)(E-p)}}{2\sqrt{2}m} \tag{19b}$$

However, $A^\dagger A + B^\dagger B = \mathbb{1}$, so the matrix $\mathcal{U}$ is unitary. Please note that this matrix acts on the *spin* indices (σ, λ), and not on the spinorial indices; it is 4×4 matrix. Alternatively, the transformation can be written:

$$u_\sigma^\alpha = \left[A_{\sigma\lambda} \otimes I_{\alpha\beta} + B_{\sigma\lambda} \otimes \gamma^5_{\alpha\beta}\right] u_\lambda^\beta, \tag{20a}$$

$$v_\sigma^\alpha = \left[A_{\sigma\lambda} \otimes I_{\alpha\beta} + B_{\sigma\lambda} \otimes \gamma^5_{\alpha\beta}\right] v_\lambda^\beta. \tag{20b}$$

We now investigate the properties of the helicity-basis 4-spinors with respect to the discrete symmetry operations P, C, and T. It is expected that $\lambda \to -\lambda$ under parity, as Berestetskiĭ, Lifshitz, and Pitaevskiĭ claimed (Berestetskiĭ *et al.*, 1982).[6] With respect to $\mathbf{p} \to -\mathbf{p}$ (i. e., the spherical system angles $\theta \to \pi - \theta$, $\varphi \to \pi + \varphi$) the helicity 2-eigenspinors transform as follows: $\phi_{\uparrow\downarrow} \Rightarrow -i\phi_{\downarrow\uparrow}$. Hence,

$$Pu_\uparrow(-\mathbf{p}) = -iu_\downarrow(\mathbf{p}), \quad Pv_\uparrow(-\mathbf{p}) = +iv_\downarrow(\mathbf{p}), \tag{21a}$$

$$Pu_\downarrow(-\mathbf{p}) = -iu_\uparrow(\mathbf{p}), \quad Pv_\downarrow(-\mathbf{p}) = +iv_\uparrow(\mathbf{p}). \tag{21b}$$

Thus, on the level of classical fields, we observe that the helicity 4-spinors transform to the 4-spinors of the opposite helicity.

[6] Indeed, if $\mathbf{x} \to -\mathbf{x}$, then the vector $\mathbf{p} \to -\mathbf{p}$, but the axial vector $\mathbf{S} \to \mathbf{S}$, that implies the above statement.

Under the charge conjugation operation we have:

$$C = \begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix} \mathcal{K}. \tag{22}$$

Hence, we observe

$$Cu_\uparrow(\mathbf{p}) = -v_\downarrow(\mathbf{p}), \quad Cv_\uparrow(\mathbf{p}) = +u_\downarrow(\mathbf{p}), \tag{23a}$$

$$Cu_\downarrow(\mathbf{p}) = +v_\uparrow(\mathbf{p}), \quad Cv_\downarrow(\mathbf{p}) = -u_\uparrow(\mathbf{p}). \tag{23b}$$

due to the properties of the Wigner operator $\Theta\phi_\uparrow^* = -\phi_\downarrow$ and $\Theta\phi_\downarrow^* = +\phi_\uparrow$. For the CP (and PC) operation we get:

$$CPu_\uparrow(-\mathbf{p}) = -PCu_\uparrow(-\mathbf{p}) = +iv_\uparrow(\mathbf{p}), \tag{24a}$$

$$CPu_\downarrow(-\mathbf{p}) = -PCu_\downarrow(-\mathbf{p}) = -iv_\downarrow(\mathbf{p}), \tag{24b}$$

$$CPv_\uparrow(-\mathbf{p}) = -PCv_\uparrow(-\mathbf{p}) = +iu_\uparrow(\mathbf{p}), \tag{24c}$$

$$CPv_\downarrow(-\mathbf{p}) = -PCv_\downarrow(-\mathbf{p}) = -iu_\downarrow(\mathbf{p}). \tag{24d}$$

Similar conclusions can be drawn in the Fock space. We define the field operator as follows:

$$\Psi(x^\mu) = \sum_\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sqrt{m}}{2E} \left[u_\lambda a_\lambda \, e^{-ip_\mu x^\mu} + v_\lambda b_\lambda^\dagger \, e^{+ip_\mu x^\mu} \right]. \tag{25}$$

The commutation relations are assumed to be the standard ones (Bogoliubov and Shirkov, 1980; Greiner, 1996; Itzykson and Zuber, 1980; Weinberg, 1995)[7] (compare with (Ahluwalia, 1996; Dvoeglazov, 1997a,b; Gupta, 1967))

$$\begin{aligned} [a_\lambda(\mathbf{p}), a_{\lambda'}^\dagger, (\mathbf{k})]_+ &= 2E\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{\lambda\lambda'}, \quad [a_\lambda(\mathbf{p}), a_{\lambda'}(\mathbf{k})]_+ = 0 \\ &= [a_\lambda^\dagger(\mathbf{p}), a_{\lambda'}^\dagger(\mathbf{k})]_+, \end{aligned} \tag{26a}$$

$$[a_\lambda(\mathbf{p}), b_\lambda^\dagger, (\mathbf{k})]_+ = 0 = [b_\lambda(\mathbf{p}), a_{\lambda'}^\dagger(\mathbf{k})]_+, \tag{26b}$$

$$\begin{aligned} [b_\lambda(\mathbf{p}), b_\lambda^\dagger, (\mathbf{k})]_+ &= 2E\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{\lambda\lambda'}, \quad [b_\lambda(\mathbf{p}), b_{\lambda'}(\mathbf{k})]_+ = 0 \\ &= [b_\lambda^\dagger(\mathbf{p}), b_{\lambda'}^\dagger(\mathbf{k})]_+. \end{aligned} \tag{26c}$$

If one defines $U_P\Psi(x^\mu)U_P^{-1} = \gamma^0\Psi(x^{\mu'})$, $U_C\Psi(x^\mu)U_C^{-1} = \tilde{C}\Psi^\dagger(x^\mu)$ and the antiunitary operator of time reversal $(V_T\Psi(x^\mu)V_T^{-1})^\dagger = T\Psi^\dagger(x^{\mu''})$, then it is easy to obtain the corresponding transformations of the creation/annihilation operators

[7] The only possible changes may be related to a different form of normalization of 4-spinors, which would have influence on the factor before δ-function.

(cf. the cited textbooks).

$$U_P a_\lambda U_p^{-1} = -i a_{-\lambda}(-\mathbf{p}), \quad U_P b_\lambda U_p^{-1} = -i b_{-\lambda}(-\mathbf{p}), \tag{27a}$$

$$U_C a_\lambda U_C^{-1} = (-1)^{\frac{1}{2}+\lambda} b_{-\lambda}(\mathbf{p}), \quad U_C b_\lambda U_C^{-1} = (-1)^{\frac{1}{2}-\lambda} a_{-\lambda}(-\mathbf{p}), \tag{27b}$$

As a consequence, we obtain (provided that $U_P|0\rangle = |0\rangle$, $U_C|0\rangle = |0\rangle$)

$$U_P a_\lambda^\dagger(\mathbf{p})|0\rangle = U_P a_\lambda^\dagger U_P^{-1}|0\rangle = i a_{-\lambda}^\dagger(-\mathbf{p})|0\rangle = i|-\mathbf{p}, -\lambda\rangle^+, \tag{28a}$$

$$U_P b_\lambda^\dagger(\mathbf{p})|0\rangle = U_P b_\lambda^\dagger U_P^{-1}|0\rangle = i b_{-\lambda}^\dagger(-\mathbf{p})|0\rangle = i|-\mathbf{p}, -\lambda\rangle^-; \tag{28b}$$

and

$$U_C a_\lambda^\dagger(\mathbf{p})|0\rangle = U_C a_\lambda^\dagger U_C^{-1}|0\rangle = (-1)^{\frac{1}{2}+\lambda} b_{-\lambda}^\dagger(\mathbf{p})|0\rangle = (-1)^{\frac{1}{2}+\lambda}|\mathbf{p}, -\lambda\rangle^-, \tag{29a}$$

$$U_C b_\lambda^\dagger(\mathbf{p})|0\rangle = U_C b_\lambda^\dagger U_C^{-1}|0\rangle = (-1)^{\frac{1}{2}-\lambda} a_{-\lambda}^\dagger(\mathbf{p})|0\rangle = (-1)^{\frac{1}{2}-\lambda}|\mathbf{p}, -\lambda\rangle^+. \tag{29b}$$

Finally, for the CP operation one should obtain:

$$\begin{aligned} U_P U_C a_\lambda^\dagger(\mathbf{p})|0\rangle &= -U_C U_P a_\lambda^\dagger(\mathbf{p})|0\rangle = (-1)^{\frac{1}{2}+\lambda} U_P b_{-\lambda}^\dagger(\mathbf{p})|0\rangle \\ &= i(-1)^{\frac{1}{2}+\lambda} b_\lambda^\dagger(-\mathbf{p})|0\rangle = i(-1)^{\frac{1}{2}+\lambda}|-\mathbf{p}, \lambda\rangle^-, \end{aligned} \tag{30a}$$

$$\begin{aligned} U_P U_C b_\lambda^\dagger(\mathbf{p})|0\rangle &= -U_C U_P b_\lambda^\dagger(\mathbf{p}) = (-1)^{\frac{1}{2}-\lambda} U_P a_{-\lambda}^\dagger(\mathbf{p})|0\rangle \\ &= i(-1)^{\frac{1}{2}-\lambda} a_\lambda^\dagger(-\mathbf{p})|0\rangle = i(-1)^{\frac{1}{2}-\lambda}|-\mathbf{p}, \lambda\rangle^+. \end{aligned} \tag{30b}$$

As in the classical case, the P and C operations anticommutes in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ quantized case. This opposes to the theory based on 4-spinor eigenstates of chiral helicity (cf. Dvoeglazov, 1997b).

Since the V_T is an antiunitary operator the problem must be solved after taking into account that in this case the c-numbers should be put outside the hermitian conjugation *without* complex conjugation:

$$\left[V_T \lambda A V_T^{-1}\right]^\dagger = \left[\lambda^* V_T A V_T^{-1}\right]^\dagger = \lambda\left[V_T A^\dagger V_T^{-1}\right]. \tag{31}$$

With this definition we obtain:[8]

$$V_T a_\lambda^\dagger V_T^{-1} = +i(-1)^{\frac{1}{2}-\lambda} a_\lambda^\dagger(-\mathbf{p}), \tag{32a}$$

$$V_T b_\lambda V_T^{-1} = +i(-1)^{\frac{1}{2}-\lambda} b_\lambda(-\mathbf{p}). \tag{32b}$$

Furthermore, we observed that the question of whether a particle and an antiparticle have the same or opposite parities depend on a phase factor in the following definition:

$$U_P \Psi(t, \mathbf{x}) U_P^{-1} = e^{i\alpha} \gamma^0 \Psi(t, -\mathbf{x}). \tag{33}$$

[8] T is chosen to be $T = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}$ in order to fulfill $T^{-1}\gamma_0^T T = \gamma_0$, $T^{-1}\gamma_i^T T = \gamma_i$, and $T^T = -T$.

Indeed, if we repeat the textbook procedure (Greiner, 1996):

$$U_P\left[\sum_\lambda\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sqrt{m}}{2E}\left(u_\lambda(\mathbf{p})\,a_\lambda(\mathbf{p})\,e^{-ip_\mu x^\mu}+v_\lambda(\mathbf{p})b_\lambda^\dagger(\mathbf{p})\,e^{+ip_\mu x^\mu}\right)\right]U_P^{-1}$$
$$=e^{i\alpha}\left[\sum_\lambda\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sqrt{m}}{2E}\left(\gamma^0u_\lambda(-\mathbf{p})a_\lambda(-\mathbf{p})\,e^{-ip_\mu x^\mu}\right.\right.$$
$$\left.\left.+\gamma^0v_\lambda(-\mathbf{p})b_\lambda^\dagger(-\mathbf{p})\,e^{+ip_\mu x^\mu}\right)\right]$$
$$=e^{i\alpha}\left[\sum_\lambda\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sqrt{m}}{2E}\left(-iu_{-\lambda}(\mathbf{p})a_\lambda(-\mathbf{p})\,e^{-ip_\mu x^\mu}\right.\right.$$
$$\left.\left.+iv_{-\lambda}(\mathbf{p})b_\lambda^\dagger(-\mathbf{p})\,e^{+ip_\mu x^\mu}\right)\right]. \tag{34}$$

Multiplying by $u_{\lambda'}(\mathbf{p})$ and $v_{\lambda'}(\mathbf{p})$ consequetively, and using the normalization conditions we obtain

$$U_Pa_\lambda U_P^{-1}=-i\,e^{i\alpha}a_{-\lambda}(-\mathbf{p}), \tag{35a}$$
$$U_Pb_\lambda^\dagger U_P^{-1}=+i\,e^{i\alpha}b_{-\lambda}^\dagger(-\mathbf{p}). \tag{35b}$$

From this, if $\alpha=\pi/2$ we obtain *opposite* parity properties of creation/annihilation operators for particles and antiparticles:

$$U_Pa_\lambda U_P^{-1}=+a_{-\lambda}(-\mathbf{p}), \tag{36a}$$
$$U_Pb_\lambda U_P^{-1}=-b_{-\lambda}(-\mathbf{p}). \tag{36b}$$

However, the difference with the Dirac case still preserves (λ transforms to $-\lambda$). As a conclusion, the question of the same (opposite) relative intrinsic parity is intrinsically related to the phase factor in (33). We find somewhat similar situation with the question of constructing the neutrino field operator (cf. with the Goldhaber-Kayser creation phase factor).

Next, we find the explicit form of the parity operator U_P and prove that it commutes with the Hamiltonian operator. We prefer to use the method described in (see above, § 10.2–10.3). It is based on the anzatz that $U_P=\exp[i\alpha\hat{A}]\exp[i\hat{B}]$ with $\hat{A}=\Sigma_s\int d^3\mathbf{p}[a_{\mathbf{p},\mathbf{s}}^\dagger a_{-\mathbf{p}s}+b_{\mathbf{p}s}^\dagger b_{-\mathbf{p}s}]$ and $\hat{B}=\Sigma_s\int d^3\mathbf{p}[\beta a_{\mathbf{p},\mathbf{s}}^\dagger a_{ps}+\gamma b_{\mathbf{p}s}^\dagger b_{\mathbf{p}s}]$. On using the known operator identity

$$e^{\hat{A}}\hat{B}\,e^{-\hat{A}}=\hat{B}+[\hat{A},\hat{B}]_-+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\ldots \tag{37}$$

and $[\hat{A},\hat{B}\hat{C}]_-=[\hat{A},\hat{B}]_+\hat{C}-\hat{B}[\hat{A},\hat{C}]_+$ one can fix the parameters α,β,γ such that satisfy the physical requirements that a Dirac particle and its anti-particle have opposite intrinsic parities.

In our case, we need to satisfy (27a), i.e., the operator should invert not only the sign of the momentum, but the sign of the helicity too. We may achieve this goal by the analogous postulate $U_P = e^{i\alpha\hat{A}}$ with

$$\hat{A} = \sum_s \int \frac{d^3\mathbf{p}}{2E}[a_\lambda^\dagger(\mathbf{p})a_{-\lambda}(-\mathbf{p}) + b_\lambda^\dagger(\mathbf{p})b_{-\lambda}(-\mathbf{p})]. \tag{38}$$

By direct verification, the Eqs. (27a) are satisfied provided that $\alpha = \pi/2$. Compare this parity operator with that given in (Greiner, 1996; Itzykson and Zuber, 1980) for Dirac fields:[9]

$$U_P = \exp\left[i\frac{\pi}{2}\int d^3\mathbf{p}\sum_s(a(\mathbf{p},s)^\dagger a(\tilde{\mathbf{p}},s) + b(\mathbf{p},s)^\dagger b(\tilde{\mathbf{p}},s)\right.$$
$$\left. - a(\mathbf{p},s)^\dagger a(\mathbf{p},s) + b(\mathbf{p},s)^\dagger b(\mathbf{p},s))\right], \quad \text{(10.69) of Greiner (1996).} \tag{39}$$

By direct verification one can also come to the conclusion that our new U_P commutes with the Hamiltonian:

$$\mathcal{H} = \int d^3\mathbf{x}\Theta^{00} = \int d^3\mathbf{k}\sum_\lambda[a_\lambda^\dagger(\mathbf{k})a_\lambda(\mathbf{k}) - b_\lambda(\mathbf{k})b_\lambda^\dagger(\mathbf{k})], \tag{40}$$

i.e.

$$[U_P, \mathcal{H}] = 0. \tag{41}$$

Alternatively, we can try to choose another set of commutation relations (Ahluwalia, 1996; Dvoeglazov, 1997b) (for the set of bi-orthonormal states), that will be the matter of future publications.

Finally, because of the fact that my recent works are related to the so-called "Bargmann-Wightman-Wigner-type" quantum field theory, I want to clarify some misunderstandings in the recent discussions. This type of theories has been first proposed by Gel'fand and Tsetlin (Gel'fand and Tsetlin, 1956). In fact, it is based on the two-dimensional representation of the inversion group, which is used when someone needs to construct a theory where C and P anticommute. They indicated applicability of this theory to the description of the set of K-mesons and possible relations to the Lee-Yang result. The comutativity/anticommutativity of the discrete symmetry operations has also been investigated by Foldy and Nigam (Nigam and Foldy, 1956). Relations of the Gel'fand–Tsetlin construct to the representations of the anti-de Sitter $SO(3, 2)$ group and the general relativity theory (including continuous and discrete transformations) have been discussed in

[9] Greiner used the following commutation relations $[a(\mathbf{p}, s), a^\dagger(\mathbf{p}', s')]_+ = [b(\mathbf{p}, s), b^\dagger(\mathbf{p}', s')]_+ = \delta^3(\mathbf{p} - \mathbf{p}')\delta_{ss'}$. One should also note that the Greiner form of the parity operator is not the only one. Itzykson and Zuber (Itzykson and Zuber, 1980) proposed another one differing by the phase factors from (10.69) of (Greiner, 1996). To find relations between those two forms of the parity operator one should apply additional rotation in the Fock space.

(Sokolik, 1957) and in subsequent papers of Sokolik. E. Wigner (Wigner, 1964) presented somewhat related results at the Istanbul School on Theoretical Physics in 1962. Later, Fushchich discussed corresponding wave equations. At last, in the paper (Ahluwalia *et al.*, 1993; Dvoeglazov, 1998) the authors called a theory where a boson and its antiboson have opposite intrinsic parities as the theory of "the Bargmann–Wightman–Wigner type." Actually, the theory presented by Ahluwalia, Goldman, and Johnson is the Dirac-like generalization of the Weinberg $2(2J+1)$-theory for the spin 1. It has already been presented in the Sankaranarayanan and Good paper of 1965 (Sankaranarayanan and Good, 1965). In Dvoeglazov (1998) (and in the previous IF-UNAM preprints of 1994) I presented a theory based on a set of 6-component Weinberg-like equations (I called them the "Weinberg doubles"). In Ahluwalia (1996) the theory in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation based on the chiral helicity 4-eigenspinors was proposed. The connection with the Foldy and Nigam consideration has been claimed. The corresponding equations have been obtained in (Dvoeglazov, 1997b) and in several less known papers. However, later we found the papers by Ziino and Barut (Barut and Ziino, 1993; Ziino, 1989, 1991, 1996) and the Markov papers (Markov, 1937, 1964), which also have connections with the subject under consideration.

A similar theory may be constructed from our consideration above if we define the field operators as follows:

$$\Psi_1 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sqrt{m}}{2E}[(u_\uparrow a_\uparrow + v_\uparrow b_\uparrow)\, e^{-ip_\mu x^\mu} + (u_\uparrow a_\uparrow^\dagger + v_\uparrow b_\uparrow^\dagger)\, e^{+ip_\mu x^\mu}], \quad (42a)$$

$$\Psi_2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sqrt{m}}{2E}[(u_\downarrow a_\downarrow - v_\downarrow b_\downarrow)\, e^{-ip_\mu x^\mu} + (u_\downarrow a_\downarrow^\dagger - v_\downarrow b_\downarrow^\dagger)\, e^{+ip_\mu x^\mu}]. \quad (42b)$$

The conclusions of my talk are

- Similarly to the $(\frac{1}{2},\frac{1}{2})$ representation, the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ field functions in the helicity basis are *not* eigenstates of the common-used parity operator; $|\mathbf{p},\lambda> \Rightarrow |-\mathbf{p},-\lambda>$ both on the classical and quantum levels. This is in accordance with the earlier consideration of Berestetskiĭ, Lifshitz, and Pitaevskiĭ.
- Helicity field functions may satisfy the ordinary Dirac equation with γs to be in the spinorial representation. Meanwhile, the chiral helicity field functions satisfy the equations of the form $\hat{p}\Psi_1 - m\Psi_2 = 0$.
- Helicity field functions can be expanded in the set of the Dirac 4-spinors by means of the matrix $\mathcal{U}^{-1}$ given in this paper. Neither A, nor B are unitary, however $A^\dagger A + B^\dagger B = \mathbb{1}$.
- P and C operations anticommute in this framework, both on the classical and quantum levels (this is opposite to the theory based on the chiral helicity eigenstates (Dvoeglazov, 1997b).

- Particle and antiparticle may have either the same or the opposite properties with respect to parity. The answer depends on the choice of the phase factor in $U_P \Psi U_P^{-1} = e^{i\alpha}\gamma^0\Psi'$; alternatively, that can be made by additional rotation U_{P_2}.
- Earlier confusions in the discussion of the Gelfand–Tsetlin–Sokolik–Nigam–Foldy–Bargmann–Wightman–Wigner-type (GTsS-NF-BWW) quantum field theory have been clarified.

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